NSC-207

UNPUBLISHED PRELIMINARY DATA

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

RE-11

A GUIDANCE-NAVIGATION SEPARATION THEOREM

by

Dr. James E. Potter

August 1964

N64-28988
(ACCESSION NUMBER)
(THRU) 1
(PAGES) 28
(CODE) 27
(NASA CR OR TMX OR AD NUMBER) CR 55341
(CATEGORY)

FACILITY FORM 602

OTS PRICE

XEROX $ 2.60 ph

MICROFILM $

EXPERIMENTAL ASTRONOMY LABORATORY

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

CAMBRIDGE 39, MASSACHUSETTS

REPORTS CONTROL No.

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ACKNOWLEDGMENT

This report was prepared under DSR Contract 5007 sponsored by the National Aeronautics and Space Administration through research grant NsG 254-62

The publication of this report does not constitute approval by the National Aeronautics and Space Administration of the findings or the conclusions contained therein It is published only for the exchange and stimulation of ideas

TABLE OF CONTENTS

ABSTRACT

28928

A Guidance-Navigation Separation Theorem.* JAMES E. POTTER, Staff Engineer, Experimental Astronomy Laboratory, Massachusetts Institute of Technology, Cambridge 39, Massachusetts.

A trajectory control system for a space vehicle may be divided into two subsystems: the navigation subsystem which filters noisy navigation measurements in order to generate estimates of the vehicle's position, velocity and mass and the guidance subsystem which uses this information to generate commands for the vehicle propulsion system in order to accomplish the assigned mission with a minimum cost. Generally, in order to obtain an optimum control system it is necessary to coordinate the design of the two subsysgems so that the guidance subsystem is as insensitive as possible to the expected navigation errors and the type of errors produced by the navigation subsystem has the least effect on guidance. However, in the somewhat idealized case when the vehicle dynamical equations are linear and the cost function is quadratic, the separation theorem states that the optimum control system is obtained when the guidance and navigation subsystems are designed separately without considering their interaction. This result was obtained for sampled data control systems by Gunkel and Joseph and is extended to continuous time control systems in the present paper. Author

*Work supported by NASA Grant NsG 254-62.

A GUIDANCE-NAVIGATION SEPARATION THEOREM

by

Dr. James E. Potter
Experimental Astronomy Laboratory
Massachusetts Institute of Technology

Section I Introduction

The subject of this paper is the design of a linear black box that receives navigation measurements as inputs and generates outputs which control the thrust magnitude and vectoring of a space vehicle propulsion system. It will be assumed that the navigation measurement process and the vehicle dynamical equations can be linearized and that it is desired to design the black box to minimize the statistical mean of a quadratic mission cost function involving fuel expended, deviation from the reference or mean trajectory and terminal error.

For the purposes of design, the overall vehicle control system is often broken up into two parts - the navigation and guidance subsystems. The navigation subsystem processes navigation measurements to obtain a running estimate of the vehicle state. Navigation measurements ordinarily consist of space sextant sightings, star-landmark tracker outputs, radar data, and accelerometer outputs. The vehicle state might consist of vehicle position, velocity, mass and navigation instrument biases. The guidance subsystem makes use of the state estimate obtained from the navigation subsystem to generate inputs to the vehicle propulsion system in order to achieve the mission with minimum cost. This terminology is somewhat prone to confusion since the terms navigation system and guidance system both are frequently used to designate the overall control system. However, the definitions given above are in common use and accurately describe the design situation.

In their Ph. D. theses, Gunkel and Joseph considered sampled data or discrete time control systems having measurements contaminated with white noise as inputs. They showed that in this case optimum guidance and navigation subsystems can be synthesized independently. The guidance subsystem is designed to minimize the mission cost function assuming perfect navigation and that there are no random trajectory disturbances due to guidance implementation errors or other causes. Navigation is accomplished by using a minimum variance (or maximum likelihood or Wiener - all terms mean about the same thing) statistical filter to obtain the vehicle state from the navigation measurements. The overall navigation - guidance black box obtained in this way is optimum for the given mission cost function. Thus, synthesis of the navigation and guidance subsystems can be carried out separately and only the design of the guidance subsystem depends on the mission cost function.[1,2]

In this paper, the Gunkel - Joseph separation theorem is proved for the continuous control case with navigation measurements contaminated with correlated noise. The dynamic programming approach employed by Gunkel and Joseph is not used explicitly although it is inherent in the present method of proof. The present analysis also applies to the sampled data case if the time integrals are replaced by sums, and yields a somewhat different proof than those of Gunkel and Joseph. In the course of the proof an interesting identity relating to optimum control is derived.

The validity of Gunkel - Joseph syntheses rests on the linearity of the vehicle dynamical equations and the navigation measurement process and the use of a quadratic mission cost function. Experience has shown that linearizing vehicle dynamical equations leads to fairly accurate results and most navigation system design depends on linearizing the navigation measurement process. If the mission cost function is not quadratic, it may still be reasonable to consider only the quadratic term in the Taylor series expansion of the cost function when carrying out a perturbation analysis about an optimum trajectory. Three instances in which a quadratic mission cost function is appropriate in space vehicle control follow.

(1) A quadratic mission cost function arises in the most straightforward way in the case of a vehicle employing electric propulsion. With constant power, variable specific impulse and constant engine effeciency, minimizing propellant expenditure corresponds to minimizing

$$C = \int_b^a \left|\underline{a}_T\right|^2 dt \tag{1-1}$$

where $\underline{a}_T$ represents thrust acceleration. A more realistic cost function is

$$C = k\left|\underline{x}_T\right|^2 + \int_a^b \left|\underline{a}_T\right|^2 dt \tag{1-2}$$

where $\underline{x}_T$ represents terminal error. This formulation allows a trade off between targeting accuracy and fuel expenditure. If k is zero in equation (1-2), the guidance system gain will become excessive in the vicinity of the target.

(2) With discrete impulsive thrusting or velocity corrections, a reasonable design approach is to try to minimize the mean of the sum of the squares of the velocity increments plus the target miss. In this case the cost function is given by

$$C = k\left|\underline{x}_T\right|^2 + \sum_{k=1}^{n} \left|\underline{v}_T\right|^2 \tag{1-3}$$

It is frequently felt that, if there is a large state of uncertainty at the time when a velocity correction is to be made, it is not desirable to make the full correction computed on the assumption that the state estimate is exact, since new velocity errors would be introduced which would later have to be corrected. This is probably true when the cost function is the total fuel expended in making the velocity corrections. However, in the case of the quadratic cost function in equation (1-3), the separation theorem implies that the best strategy is, regardless of the actual state uncertainty, to make the full velocity correction at each correction time as if there were no state uncertainty.

(3) In the chemical powered segments of a space mission the goal is often to follow a reference trajectory which is optimum in some sense. The function of the control system in this case is to minimize the deviations from the nominal thrust program and the reference trajectory due to nonstandard initial conditions and vehicle parameters. A suitable cost function for this situation is

$$C = k_1 \underline{x}_T{}^2 + k_2 \int_a^b |\underline{x}|^2 \, dt + \int_a^b |\underline{u}|^2 \, dt$$

Here, $\underline{x}$ denotes the deviation from the state the vehicle would have on the reference trajectory and $\underline{u}$ denotes the deviation from the nominal thrust program.

The separation theorem also applies to attitude control system design and instrument design or other control problems where a linear dynamical system model and a quadratic cost function are appropriate.

It will be assumed that the behavior of the dynamical system is described by the linear vector differential equation

$$\underline{x}' = F(t)\,\underline{x} + G(t)\,\underline{u} + \underline{\nu} \tag{1-4}$$

In this equation, the k-dimensional vector $\underline{x}$ represents the output or state of the dynamical system, the p-dimensional vector $\underline{u}$ represents the control input to the dynamical system and the k-dimensional vector $\underline{\nu}$ represents the disturbing forces driving the dynamical system. F(t) and G(t) denote k by k and k by p matrices respectively. The mission or control problem will be assumed to start at time $t = 0$ and end at time $t = f$.

The object in designing the feedback loop which generates the dynamical system control input u using information about its output x is to minimize the quadratic mission cost function

$$C = \underline{x}^T(f)\, C_1\, \underline{x}(f) + \int_0^f \left\{ \underline{u}^T(t)\, C_2(t)\, \underline{u}(t) + \underline{x}^T(t)\, C_3(t)\, \underline{x}(t) \right\} dt \tag{1-5}$$

In this cost function, the k by k matrix C_1 determines the weighting to be given to terminal error, the p by p matrix $C_2(t)$ weights the magnitude of the corrective signal generated by the feedback loop and the k by k matrix $C_3(t)$ weights the error in state during the mission. The matrices C_1, $C_2(t)$ and $C_3(t)$ are assumed to be positive definite and symmetric. The term $\underline{u}^T C_2 \underline{u}$ is included in the cost function either to economize on fuel used to control the system, as would be the case in making velocity corrections to control the trajectory of a space vehicle, or to keep the control input within the linear response range of the dynamical system. If this term is not included in the cost function, the mathematically optimum feedback loop has infinite gain and nulls out the state error by sending a delta function control signal to the dynamical system as soon as the system is turned on. This kind of performance is generally not what one is aiming for in a real system and it is therefore necessary to include the $\underline{u}^T C_2 \underline{u}$ term in the cost function.

If the mission is already partially completed, C(t) given by the formula

$$C(t) = \underline{x}^T(f)\, C_1\, \underline{x}(f) + \int_t^f \left\{ \underline{u}^T C_2 \underline{u} + \underline{x}^T C_3\, \underline{x} \right\} dt \qquad (1\text{-}6)$$

will denote the cost of completing the mission from time t onward.

The signals available for generating the control signal $\underline{u}$ will be assumed to be of the form

$$\underline{m}(t) = H(t)\, \underline{x}(t) + \underline{n}(t) \qquad (1\text{-}7)$$

Here $\underline{m}$ is a q-dimensional vector and H is a q by k matrix and $\underline{n}(t)$ represents measurement noise. In order for the separation theorem to hold, all that needs to be assumed about the statistics of $\underline{n}$ is that $\underline{n}$ has mean zero and is uncorrelated with the disturbing force $\underline{\nu}$.

The following notation will be used to describe the filters employed in the feedback loop.

$$\underline{u}(t) = \tilde{L}\left[\underline{m}\right](t)$$

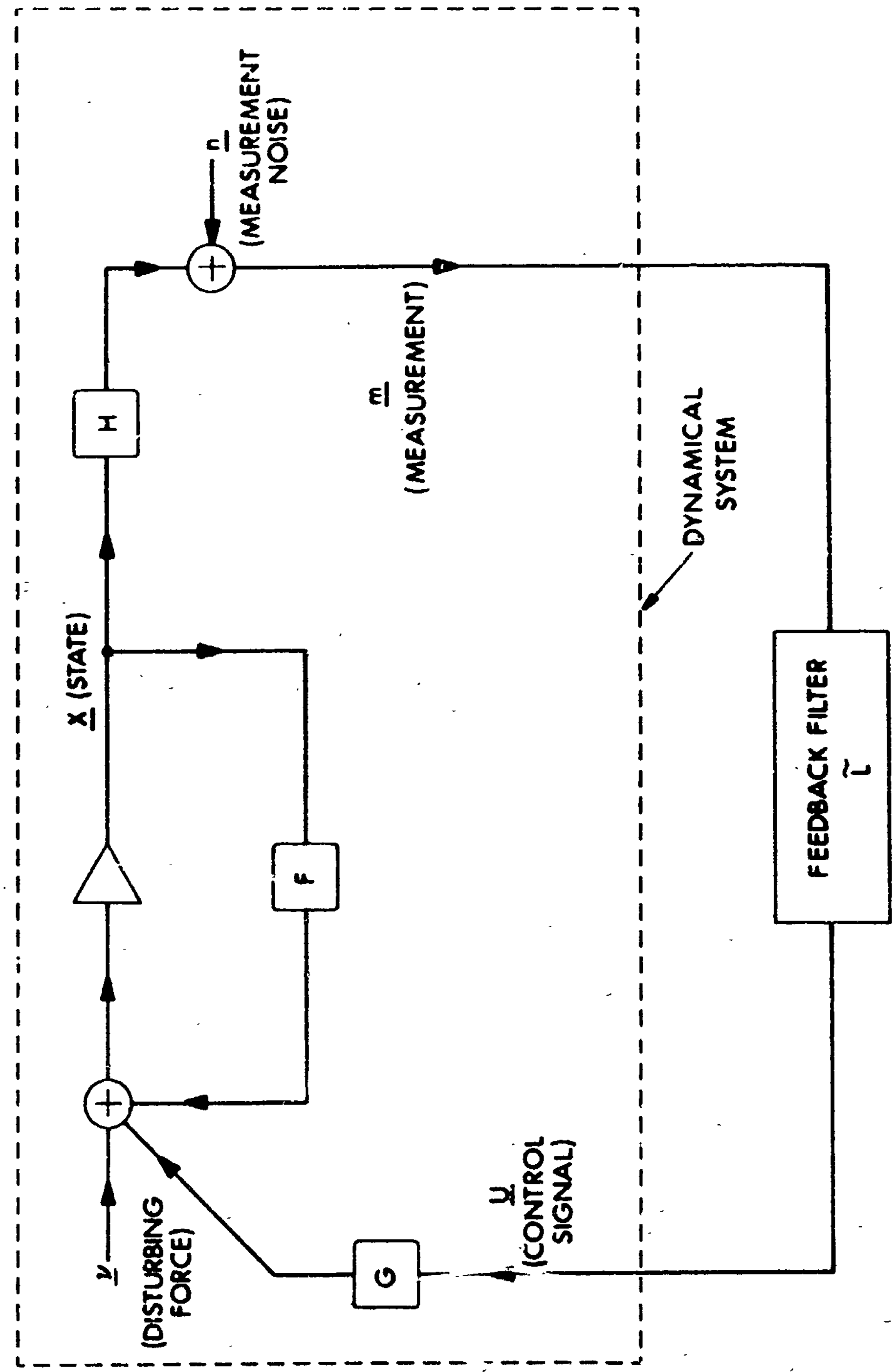
$\tilde{L}$ denotes the filter. The tilde (~) is used as a reminder that $\tilde{L}$ is not just a gain matrix but represents a black box containing integrators, internal feedback loops, etc. The quantity $\underline{m}$ inside the square brackets denotes the input signal to the filter. Thus $\tilde{L}\left[\underline{a}\right](t)$ denotes the output of filter $\tilde{L}$ at time t when the input is the signal $\underline{a}$.

Figure 1 is a signal flow diagram of the dynamical system and filter $\tilde{L}$.

Section 2 Minimum Variance Open Loop Estimation

Throughout this section it will be assumed that the dynamical system is operated without feedback, that is $\underline{u} \equiv 0$. The problem of estimating the state of a linear dynamical system driven by white noise on the basis of noisy measurements has been extensively treated in the literature. The purpose of this section is to derive two properties of the

FIGURE I


ν
(DISTURBING FORCE)
G
u
(CONTROL SIGNAL)
F
X (STATE)
H
n
(MEASUREMENT NOISE)
m
(MEASUREMENT)
DYNAMICAL SYSTEM
FEEDBACK FILTER
L̃

optimum linear estimator which are needed in the proof of the separation theorem.

If ξ is a random vector with mean zero, a generalized variance of ξ will be defined as

$$\mathrm{Var}_Q(\underline{\xi}) = \overline{\underline{\xi}^T Q \underline{\xi}}$$

where Q is a positive semidefinite matrix. If Q is positive definite and not merely semidefinite, Var_Q will be called a positive definite generalized variance. The generalized variance is an extension to vectors of the idea of an rms value. For example, if Q is the identity matrix, $\mathrm{Var}_I(\underline{\xi})$ is the mean square length of the random vector $\underline{\xi}$.

The filter $\tilde{L}_{MV}$ will be called a minimum variance estimator of the state $\underline{x}$ if, for each time t and every Q, the generalized variance of the estimation error, $\mathrm{Var}_Q\left(\underline{x}(t) - \tilde{L}_{MV}\left[\underline{m}\right](t)\right)$, is less than the corresponding generalized variance for any other linear filter applied to the measurements, $\underline{m}$.

In an optimization problem one expects to be asked to minimize a single cost function, in this case a particular generalized variance of the estimation error. If two or more cost functions are involved, a solution which is optimum for one cost function would not be likely to be optimum for the others and one would not expect to be able to simultaneously minimize all of the cost functions. Therefore, one might wonder whether a minimum variance estimator as defined above ever exists. In this respect the optimum statistical estimation problem is quite unusual, for the following result holds:

> Remark 1 - If at time t, the filter $\tilde{L}$ minimizes a given positive definite generalized variance of the estimation error, $\underline{x}(t) - \tilde{L}\left[\underline{m}\right](t)$, then it minimizes every generalized variance of the estimation error at that time. Thus, a filter designed to minimize a given positive definite generalized variance of the estimation error also minimizes every other generalized variance.

This remark is the result of the fact that regardless of which positive definite generalized variance is used as cost function, the criterion for an optimum filter is that the estimation error be uncorrelated with the past measurements. That is

$$\overline{\left(\underline{x}(t) - \tilde{L}\left[\underline{m}\right](t)\right)\underline{m}^T(s)} = 0$$

for $s \leqq t$.

Under quite general assumptions about the measurement noise $\underline{n}(t)$, Weiner[3], Kalman[4], Battin[5], Bryson[6], and Deyst[7] have derived filters that minimize some generalized variance of the estimation error (e.g., Battin minimizes the mean square length of the estimation error vector).

By the remark above, it follows that these are minimum variance estimators. It will be assumed that a minimum variance estimator $\tilde{L}_0$ for the state of the open loop dynamical system can be obtained by one of these methods.

The second remark about optimum open loop estimators concerns the minimum variance estimator of a vector $\underline{y}(t)$ which results from multiplying the state $\underline{x}(t)$ by a time varying matrix $M(t)$. In this case the following nearly obvious remark applies:

Remark 2 - If

$$\underline{z}(t) = M(t)\,\underline{x}(t)$$

and $\tilde{L}_0[\underline{m}](t)$ is the minimum variance estimator of $\underline{x}(t)$, then $M(t)\tilde{L}_0[\underline{m}](t)$ is the minimum variance estimator of $\underline{z}(t)$.

Remark 2 may be verified as follows. The correlation of the $\underline{y}(t)$ estimation error with the past measurements is given by the expression

$$\overline{\left(\underline{y}(t) - M(t)\tilde{L}_{ML}[\underline{m}](t)\right)\underline{m}^T(t)} - M(t)\left\{\overline{\left(\underline{x}(t) - \tilde{L}_{ML}[\underline{m}](t)\right)\underline{m}^T(t)}\right\}$$

The quantity in brackets on the right hand side of the equation above vanishes since $\tilde{L}_{ML}$ is the minimum variance estimator of $\underline{x}$. Thus, it follows that

$$\overline{\left(\underline{y}(t) - M(t)\tilde{L}_{ML}[\underline{m}](t)\right)\underline{m}^T(t)} = 0$$

and therefore that $M\tilde{L}_{ML}[\underline{m}]$ is the minimum variance estimator of $\underline{y}$.

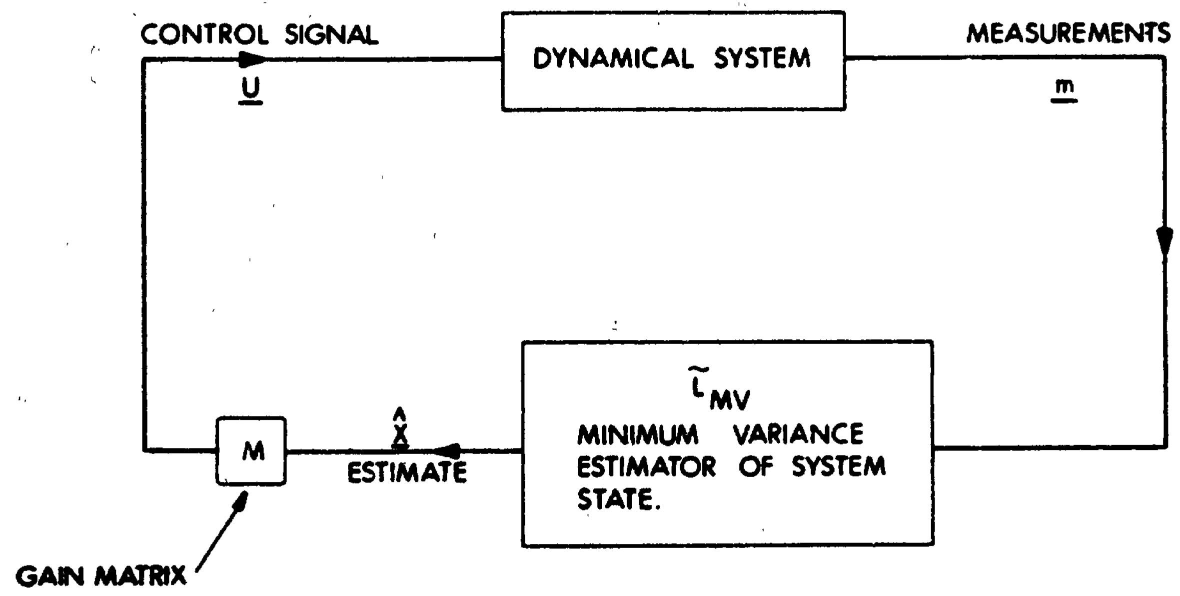
Section 3 Synthesis of the Minimum Variance Estimator in a Closed Loop System

As indicated in Section 1, the optimum external feedback loop for controlling the dynamical system will turn out to be a minimum variance estimator of the system state $\underline{x}$ followed by a gain matrix, M. This gain matrix is designed on the assumption that the state estimate is exact. This system configuration is illustrated in Fig. 2.

This section is concerned with the filter $\tilde{L}_{MV}$. It will be assumed in this section that the gain matrix $M(t)$ has already been chosen.

At first glance, design of a closed loop minimum variance estimator seems essentially different from the design of an open loop estimator since the output of the estimator effects the future states of the dynamical system. Thus, the estimator has some control over the future measurements and it might be wise not to estimate accurately at the present time in order to create a favorable estimation environment in the future. This would indeed be the case with a nonlinear system. Luckily, with a linear system, superposition allows the effects of feedback to be eliminated since the estimator knows without any uncertainty what the control signal $\underline{u}$ is. By means of feedback within the estimation filter the effects of the known control signals can be removed and the statis-

FIGURE 2



CONTROL SIGNAL
U
DYNAMICAL SYSTEM
MEASUREMENTS
m
$\tilde{L}_{MV}$
MINIMUM VARIANCE ESTIMATOR OF SYSTEM STATE.
$\hat{X}$
ESTIMATE
M
GAIN MATRIX

tical filtering can be carried out by open loop techniques.

Figure 3 illustrates the construction of a closed loop estimator by adding extra feedback loops to an open loop estimator. In practice, the open loop estimator $\tilde{L}_0$ usually also contains a dynamical system model which may be combined with the one shown in Fig. 3 to obtain a somewhat less complex overall feedback loop.

The following calculations show that the estimation error statistics for the open and closed loop estimators are the same. From Fig. 3 it follows that

$$\underline{x}' = F\,\underline{x} + G\,\underline{u} + \underline{\nu} \tag{3-1}$$

and

$$\underline{z}' = F\,\underline{z} + G\,\underline{u} \tag{3-2}$$

Letting $\underline{y} = \underline{x} - \underline{z}$ and subtracting Eq. (3-2) from Eq. (3-1) yields

$$\underline{y}' = F\,\underline{y} + \underline{\nu} \tag{3-3}$$

Also, from Fig. 3

$$\underline{m}_1 = H\,\underline{x} - H\,\underline{z} + \underline{n}$$

or

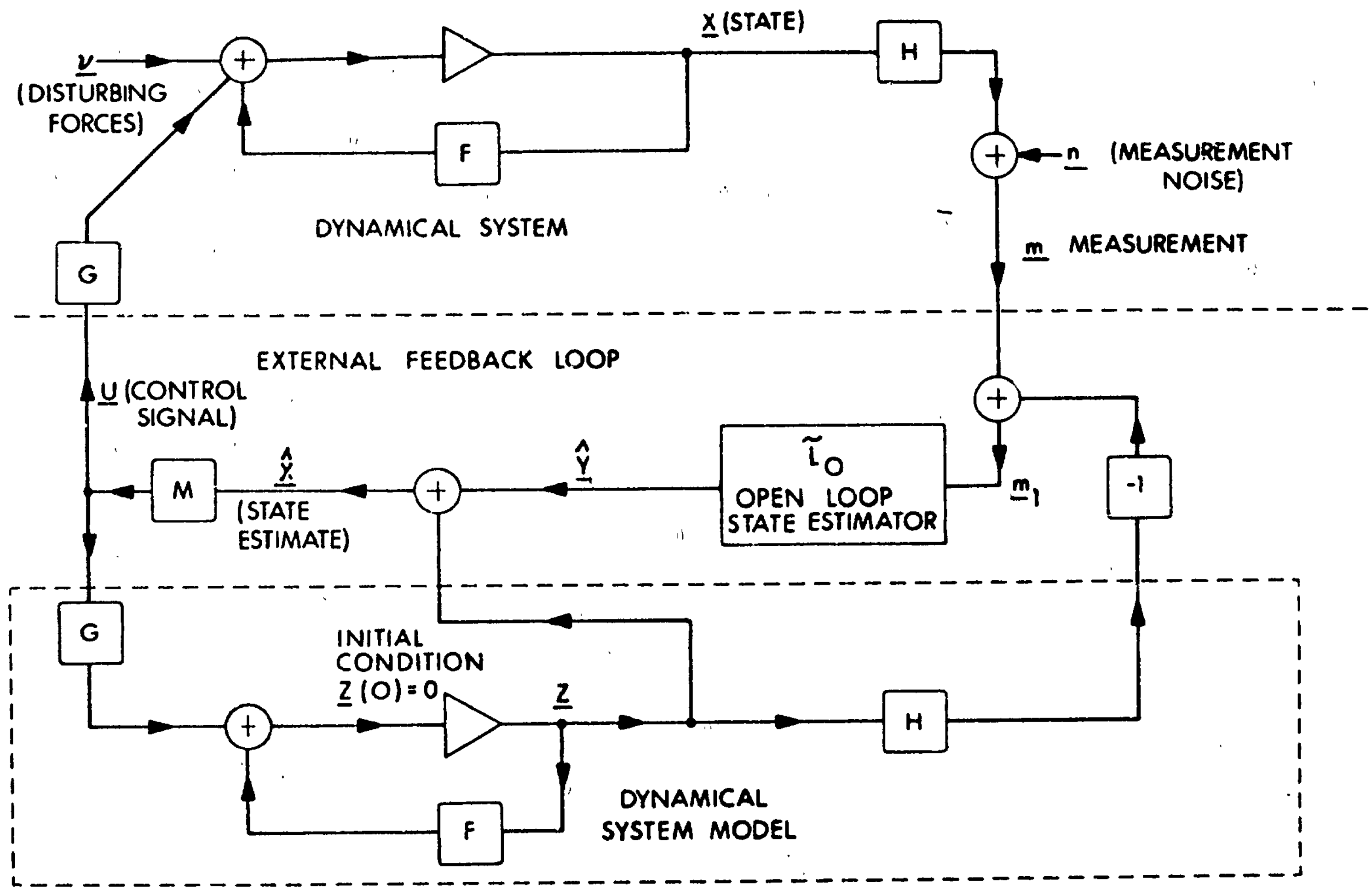
$$\underline{m}_1 = H\,\underline{y} + \underline{n} \tag{3-4}$$

Furthermore, since $\underline{z}(0) = 0$ it follows that $\underline{y}(0) = \underline{x}(0)$ and the statistics of $\underline{y}(0)$ are the same as those of $\underline{x}(0)$. Since the input to $\tilde{L}_0$ is $\underline{m}_1$, it follows from Eqs. (3-3) and (3-4) that the output $\underline{y}$ of $\tilde{L}_0$ is an open loop estimate of $\underline{y}$. Thus the random vector $(\underline{y} - \hat{\underline{y}})$ has open loop estimation error statistics. But $\underline{y} = \underline{x} - \underline{z}$ and $\hat{\underline{y}} = \hat{\underline{x}} - \underline{z}$ from Fig. 3. Thus $\underline{y} - \hat{\underline{y}} = \underline{x} - \hat{\underline{x}}$ and hence $(\underline{x} - \hat{\underline{x}})$ has the same statistics as the open loop estimation error.

The transition from a closed loop estimator to an open loop estimator is not of much practical interest. However, it is necessary to the theory in order to establish the fact that there is a minimum variance closed loop estimator. Figure 4 shows how a closed loop estimator may be used for open loop estimation by providing simulated control action through an added feedback loop. By calculations similar to those carried out above it can be shown that the estimation error statistics for this filter are the same as they would have been with closed loop estimation.

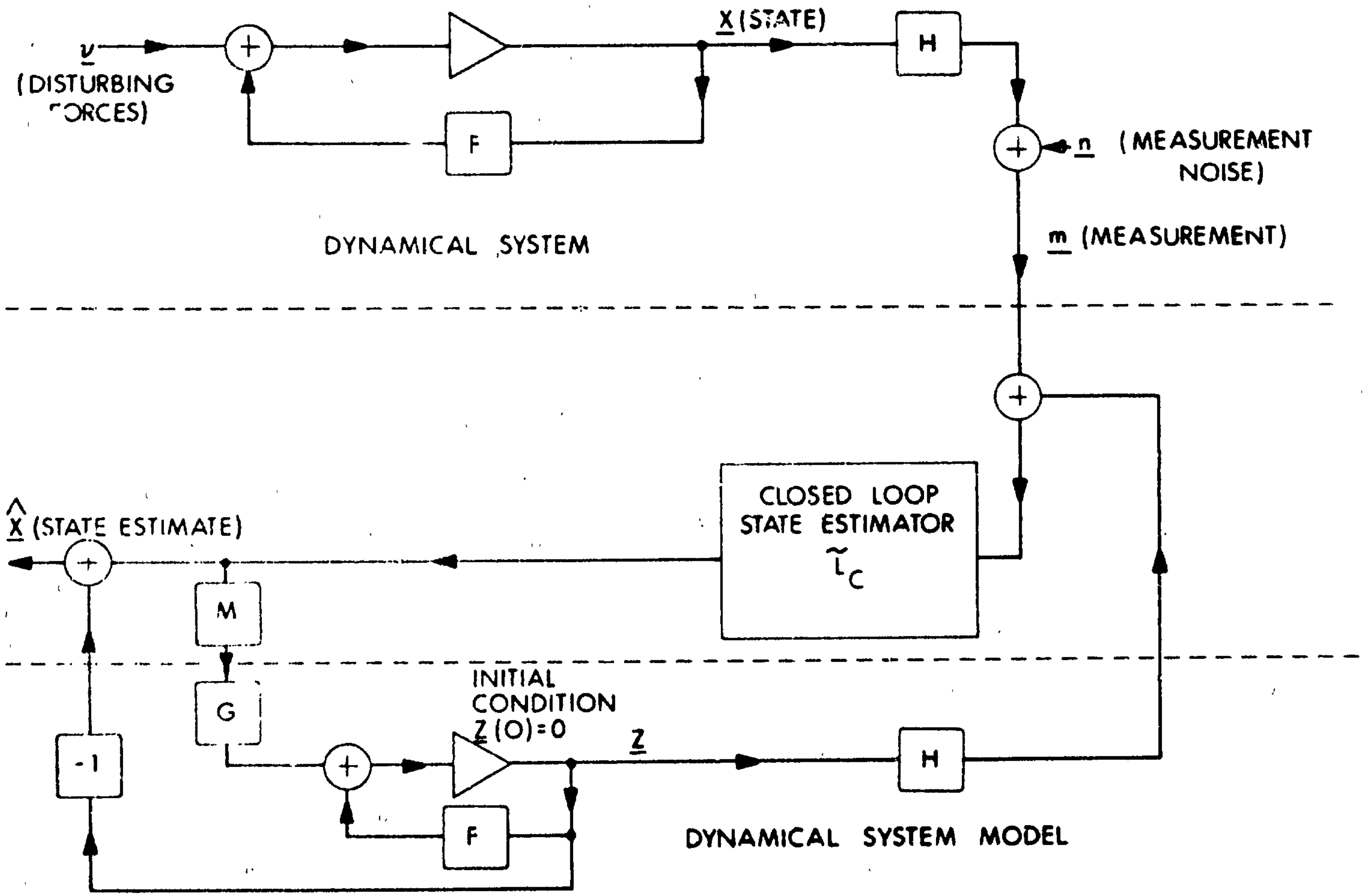
It now follows that, if $\tilde{L}_0$ is the open loop minimum variance estimator of the state of the dynamical system, then the closed loop estimator $\tilde{L}_c$ shown in Fig. 3 is the minimum variance closed loop estimator. For, suppose there was a closed loop estimator $\tilde{L}_c^{(1)}$ which at time t made some generalized variance of the estimation error smaller than the same generalized variance of the estimation error using $\tilde{L}_c$. It was

FIGURE 3



ν
(DISTURBING
FORCES)
X (STATE)
H
F
G
n (MEASUREMENT
NOISE)
m MEASUREMENT
DYNAMICAL SYSTEM
EXTERNAL FEEDBACK LOOP
U (CONTROL
SIGNAL)
M
X̂
(STATE
ESTIMATE)
Ŷ
L̃O
OPEN LOOP
STATE ESTIMATOR
m1
-1
G
INITIAL
CONDITION
Z (O) = 0
Z
H
F
DYNAMICAL
SYSTEM MODEL

FIGURE 4



ν
(DISTURBING
FORCES)
X (STATE)
H
F
n (MEASUREMENT
NOISE)
m (MEASUREMENT)
DYNAMICAL SYSTEM
CLOSED LOOP
STATE ESTIMATOR
$\tilde{L}_C$
$\hat{X}$ (STATE ESTIMATE)
M
G
-1
INITIAL
CONDITION
Z(O)=0
Z
DYNAMICAL SYSTEM MODEL

shown above that the closed loop estimator $\tilde{L}_c^{(1)}$ can be converted to an open loop estimator $\tilde{L}_0^{(1)}$ with the same estimation error statistics. Finally, this would mean that $\tilde{L}_0^{(1)}$ is a better open loop filter at time t than $\tilde{L}_0$ is. This is a contradiction since $\tilde{L}_0$ is the minimum variance estimator.

An equivalent approach to closed loop estimator design is to synthesize a minimum variance estimator

$$\hat{\underline{x}} = \tilde{L}\left[\underline{m}, \underline{u}\right]$$

assuming that the control program $\underline{u}(t)$ is known a priori. To use this filter as a closed loop estimator the signal $M\hat{\underline{x}}$ is fed back to the $\underline{u}$ input of the filter. It is not obvious that this procedure always makes sense mathematically. However, in most cases of practical interest, the equation

$$\hat{\underline{x}} = \tilde{L}\left[\underline{m}, M\hat{\underline{x}}\right]$$

determines $\hat{\underline{x}}$ as a function of $\underline{m}$ and the estimation error staristics are the same as in the case when $\underline{u}(t)$ is really known a priori.

The fact which is needed in the proof of the separation theorem is that $\underline{u} = M\hat{\underline{x}}$ is the minimum variance closed loop estimator of the optimum control signal $M\underline{x}$. This follows easily from the remarks above and the fact that in the open loop case $M\hat{\underline{x}}$ is the minimum variance estimator of $M\underline{x}$.

Example - If the measurement noise $\underline{n}(t)$ is white and uncorrelated with the initial state of the system, Kalman showed that the open loop minimum variance estimator of the dynamical system state is the filter illustrated in Fig. 5 In this filter, the weighting matrix W(t) is given by the formula

$$W(t) = N^{-1}(t)\, E(t)\, H^T(t)$$

where

$$\overline{\underline{n}(t)\, \underline{n}^T(s)} = N(t)\, \delta\,(t - s).$$

and E(t) is estimation error covariance matrix found by solving the matrix Ricatti differential equation

$$E' = F E + E F^T + S - E H^T N^{-1} H E$$

where the matrix S(t) is defined by the equation

$$\overline{\underline{\nu}(t)\, \underline{\nu}^T(s)} = S(t)\, \delta\,(t - s) \quad .$$

Substituting Fig. 5 for $\tilde{L}_0$ in Fig. 3 yields the filter illustrated in Fig. 6. This filter may be simplified to yield the form illustrated in Fig. 7.

If the control input $\underline{u}(t)$ is known a priori, the open loop estimator shown in Fig. 5 takes on the modified form illustrated in Fig. 8. By connecting the $\underline{u}$ input of this filter to the $\hat{\underline{x}}$ output through the gain matrix M, the filter shown in Fig. 7 is again obtained.

FIGURE 5

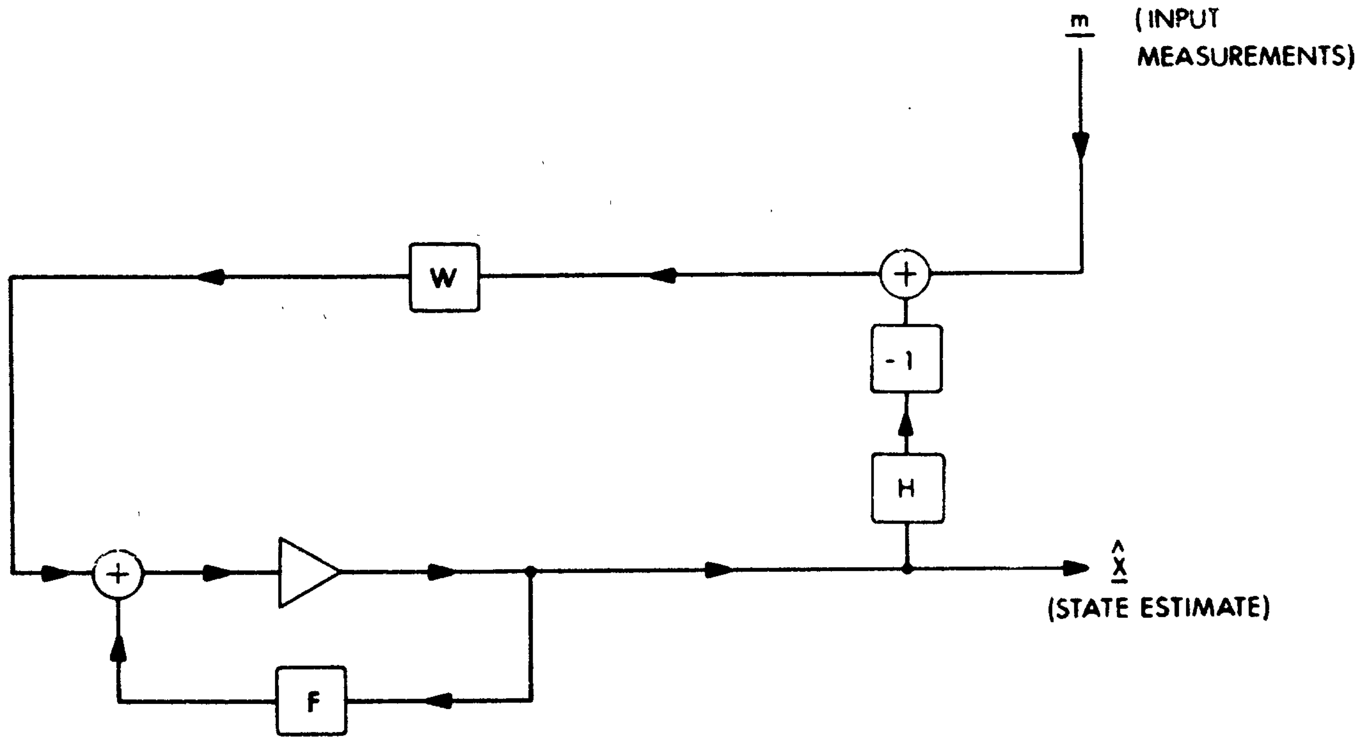
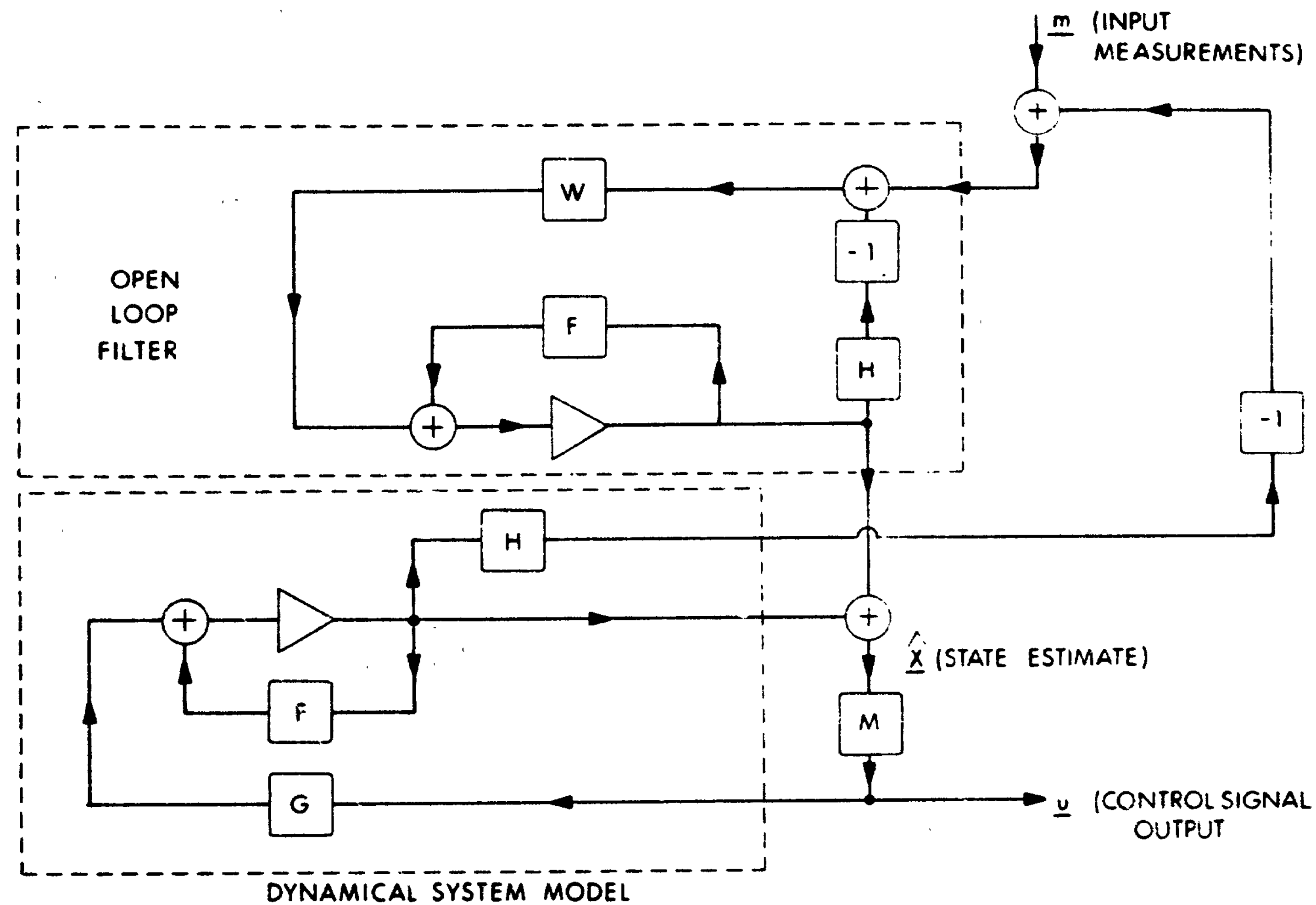


FIGURE 6


m (INPUT MEASUREMENTS)
W
-1
OPEN LOOP FILTER
F
H
-1
H
$\hat{X}$ (STATE ESTIMATE)
F
M
G
u (CONTROL SIGNAL OUTPUT)
DYNAMICAL SYSTEM MODEL

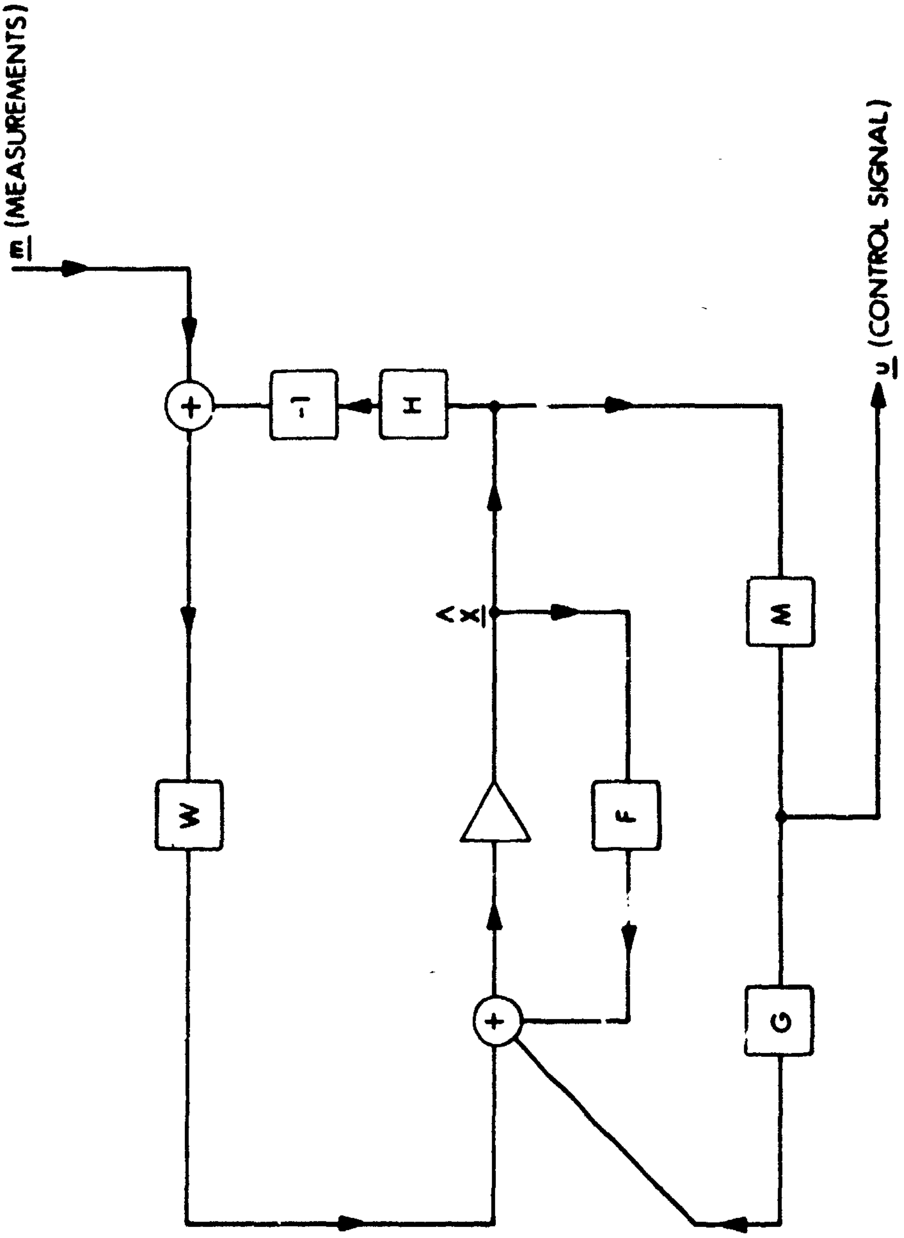
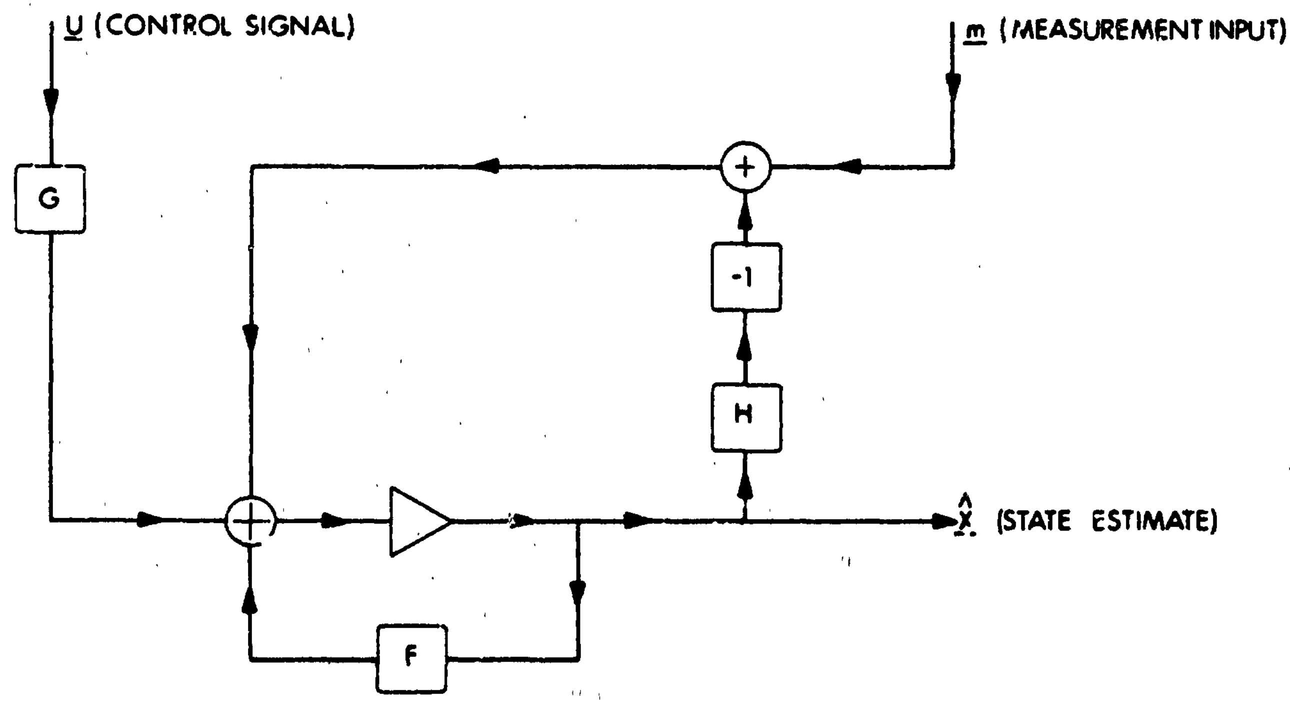


FIGURE 7

FIGURE 8



U (CONTROL SIGNAL)
m (MEASUREMENT INPUT)
G
+
-1
H
F
$\hat{x}$ (STATE ESTIMATE)

Section 4 The Fundamental Identity

Kalman[8] has shown that the control program

$$u(t) = -C_2^{-1}(t) G^T(t) U(t) x(t) \qquad (4-1)$$

minimizes the mission cost C provided that there are no disturbing forces ν and that the state is known exactly all of the time. The matrix U(t) in Eq. (4-1) is defined as the solution of the matrix Riccati differential equation

$$U' = -F^T U - U F + U G C_2^{-1} G^T U - C_3 \qquad (4-2)$$

with the boundary condition

$$U(f) = C_1 \qquad (4-3)$$

It turns out that U(t) is a positive semidefinite symmetric matrix. The physical significance of U(t) is that, if the present state of the dynamical system is $\underline{x}(t)$, the cost of completing the mission employing the optimum control program is given by the formula

$$C(t) = \underline{x}^T(t) U(t) \underline{x}(t) \qquad (4-4)$$

Equation (4-4) follows from the identity proved later in this section.

A striking feature of Kalman's result is that the feedback gain matrix, $-C_2^{-1} G^T U$, does not depend on the initial state of the dynamical system. A priori, all that one would expect to be able to accomplish, in synthesizing a linear feedback loop, would be to minimize the mean cost function for an ensemble of representative initial states. In fact, however, if there is no state uncertainty the actual cost and not its ensemble average can be minimized for every initial state employing the same feedback gain matrix.

Since, in the problems under investigation on this paper, complete information about the present state is not available to the controller, the control input to the dynamical system will be written as

$$\underline{u} = -C_2^{-1} G^T U \underline{x} + \underline{\mu} \qquad (4-5)$$

where $\underline{\mu}$ represents the deviation from the optimum control input.

In order to analyze the effects of uncertainty in the knowledge of the state and random forces driving the controlled system, a formula for the cost C(t) of completing the mission is needed.

Identity:

$$C(t) = \underline{x}^T(t) U(t) \underline{x}(t) + \underline{v}^T(t) \underline{x}(t) + w(t) \qquad (4-6)$$

where U(t) is defined by Eqs. (4-2) and (4-3) and $\underline{v}$ and w satisfy the differential equations

$$\underline{v}' = -\left\{F - G C_2^{-1} G^T U\right\}^T \underline{v} - 2 U \underline{\nu} \qquad (4-7)$$

$$w' = -\underline{v}^T G \underline{\mu} - \underline{\mu}^T A \underline{\mu} - \underline{\nu}^T \underline{v} \qquad (4-8)$$

and the boundary conditions

$$\underline{v}(f) = 0 \qquad (4-9)$$

$$w(f) = 0 \qquad (4-10)$$

Proof of Identity - From the boundary conditions imposed on U, $\underline{v}$, and w, it follows that C(f) as defined by the identity has the value

$$C(f) = \underline{x}^T(f)\, C_1\, \underline{x}(f) \tag{4-11}$$

It will be shown below that

$$\frac{dC}{dt} = -\underline{u}^T C_2 \underline{u} - \underline{x}^T C_3 \underline{x} \tag{4-12}$$

Integrating Eq. (4-12) from t to f yields

$$C(f) - C(t) = -\int_t^f (\underline{u}^T C_2 \underline{u} + \underline{x}^T C_3 \underline{x})\, dt$$

and by Eq. (4-11)

$$C(t) = \underline{x}_f^T B \underline{x}_f^T + \int_t^f (\underline{u}^T C_2 \underline{u} + \underline{x}^T C_3 \underline{x})\, dt$$

which is the correct formula for the cost to complete the mission.

It only remains to verify Eq. (4-12). Now

$$(\underline{x}^T U \underline{x})' = 2(\underline{x}')^T U \underline{x} + \underline{x}^T U' \underline{x} \tag{4-13}$$

Substituting Eqs. (1-4), (4-2) and (4-5) into (4-13) and collecting terms yields

$$(\underline{x}^T U \underline{x})' = -\underline{x}^T \left\{ U G C_2^{-1} G^T U + C_3 \right\} \underline{x} + 2(G\underline{\mu} + \underline{\nu})^T U \underline{x} \tag{4-14}$$

Similarly

$$(\underline{v}^T \underline{x})' = (\underline{v}')^T \underline{x} + \underline{v}^T \underline{x}'$$

and by (1-1), (4-2), and (4-7)

$$(\underline{v}^T \underline{x})' = -2\underline{\nu}^T U \underline{x} + \underline{v}^T (G\underline{\mu} + \underline{\nu}) \tag{4-15}$$

Adding Eqs. (4-9), (4-14), and (4-15) yields

$$C' = -\underline{x}^T \left\{ U G C_2^{-1} G^T U + C_3 \right\} \underline{x} + 2\underline{\mu}^T G^T U \underline{x} - \underline{\mu}^T C_2 \underline{\mu}$$

or collecting terms

$$C' = -\underline{x}^T C_3 \underline{x} - (-C_2^{-1} G^T U \underline{x} + \underline{\mu})^T C_2 (-C_2^{-1} G^T U \underline{x} + \underline{\mu})$$

Finally by Eq. (4-5)

$$C' = -\underline{x}^T C_3 \underline{x} - \underline{\mu}^T C_2 \underline{\mu}$$

and the proof of the identity is complete.

Note that statistics have not yet entered the picture. Two significant results can be obtained from the identity.

Result 1 - If the state $\underline{x}$ can be measured exactly and there are no disturbing forces driving the controlled system ($\underline{\nu} \equiv 0$), then

$$C(t) = \underline{x}^T(t)\, U(t)\, \underline{x}(t) + \int_t^f \underline{\mu}^T(t)\, C_2(t)\, \underline{\mu}(t)\, dt \tag{4-16}$$

This gives another proof that Kalman's feedback matrix is optimum in the deterministic case, since the cost is clearly minimized when $\underline{\mu}(t) \equiv 0$. Also, with $\underline{\mu}(t) \equiv 0$, Eq. (4-16) reduces to Eq. (4-4) and the latter equation is verified.

This result may be proved as follows. Since $\underline{\nu} \equiv 0$, Eq. (4-7) becomes

$$\underline{v}' = -\left\{F - G C_2^{-1} G^T U\right\}^T \underline{v}$$

with the boundary condition

$$\underline{v}(f) = 0$$

In this case $\underline{v}(t) \equiv 0$ since it is the solution of a homogeneous linear differential equation with a zero boundary condition. With $\underline{\nu} = \underline{v} = 0$, Eq. (4-8) becomes

$$w' = -\underline{\mu}^T C_2 \underline{\mu} \tag{4-17}$$

with $w(f) = 0$. Thus

$$w(t) = \int_t^f \underline{\mu}^T C_2 \underline{\mu}\, dt \tag{4-18}$$

Substituting $\underline{v} = 0$ and (4-18) into the identity yields Eq. (4-16).

Result 2 - If the disturbing force $\underline{\nu}(t)$ is white noise with mean zero and covariance matrix

$$\overline{\underline{\nu}(t)\, \underline{\nu}^T(s)} = S(t)\, \delta\, (t - S) \tag{4-19}$$

and $\underline{\mu}(t)$ and $\underline{x}(t)$ are uncorrelated with $\underline{\nu}(s)$ for $t \leq s$ then the mean cost to complete the mission is given by the formula

$$\overline{C(t)} = \overline{\underline{x}^T(t)\, U(t)\, \underline{x}(t)} + \int_t^f \left\{ \overline{\underline{\mu}^T C_2 \underline{\mu}} + \mathrm{tr}\,(S U) \right\} dt$$

Proof of Result · From Eq. (4-7) it follows that

$$\underline{v}(t) = \int_t^f 2\, \Phi\,(t, s)\, U(s)\, \underline{\nu}\,(s)\, ds \qquad (4\text{-}21)$$

where Φ is the solution of the differential equation

$$\frac{\partial \Phi}{\partial t} = - \left\{ F - G\, C_2^{-1} G^T U \right\}^T \Phi$$

with the boundary condition $\Phi\,(t, t) = I$. The mean value of the second term in the identity is thus given by the equation

$$\overline{\underline{v}^T(t)\, \underline{x}(t)} = 2 \int_t^f \overline{\underline{x}^T(t) \Phi\,(t, s)\, U(s)\, \underline{\nu}\,(s)}\, ds$$

$$= 2\, \mathrm{tr} \left\{ \int_t^f \Phi(t, s)\, U(s)\, \overline{\underline{\nu}\,(s)\, \underline{x}^T(t)}\, ds \right\}$$

The matrix $\overline{\underline{\nu}\,(s)\, \underline{x}^T(t)}$ is zero since $\underline{x}^T$ is uncorrelated with $\underline{\nu}\,(s)$ for $t \leq s$ and therefore

$$\overline{\underline{v}^T(t)\, \underline{x}(t)} = 0 \qquad (4\text{-}22)$$

From equations (4-8) and (4-10) it follows that the mean of the third term of the identity is given by the formula

$$\overline{w(t)} = \int_t^f \left\{ \overline{\underline{v}^T G \underline{\mu}} + \overline{\underline{\mu}^T C_2 \underline{\mu}} + \overline{\underline{\nu}^T \underline{v}} \right\} dt \qquad (4\text{-}23)$$

By means of Eq. (4-21) the first term under the integral sign in Eq. (4-23) may be written as

$$\overline{\underline{v}^T(t)\, G(t)\, \underline{\mu}(t)} = 2\, \mathrm{tr} \left\{ \int_t^f G^T(t)\, \Phi\,(t, s)\, U(s)\, \overline{\underline{\nu}(s)\, \underline{\mu}^T(t)}\, ds \right\}$$

Since $\underline{\mu}\,(t)$ is uncorrelated with $\underline{\nu}\,(s)$ for $t \leq s$, $\overline{\underline{\nu}\,(s)\, \underline{\mu}^T(t)} = 0$ and

$$\overline{\underline{\nu}^T(t)\, G(t)\, \underline{\mu}(t)} = 0 \tag{4-24}$$

By means of Eq. (4-21) the third term under the integral sign in Eq. (4-23) may be written as

$$\overline{\underline{\nu}^T(t)\, \underline{\nu}(t)} = 2\, \mathrm{tr} \left\{ \int_t^f \Phi(t, s)\, U(s)\, \overline{\underline{\nu}(s)\, \underline{\nu}^T(t)}\, ds \right\}$$

and therefore

$$\int_{t_0}^f \overline{\nu^T(t)\, \nu(t)}\, dt = 2\, \mathrm{tr} \left\{ \int_{t_0}^f \int_t^f \Phi(t, s)\, U(s)\, \overline{\underline{\nu}(s)\, \underline{\nu}^T(t)}\, ds\, dt \right\} \tag{4-25}$$

Interchanging the order of integration on the right hand side above yields

$$\int_{t_0}^f \int_t^f \Phi(t, s)\, U(s)\, \overline{\underline{\nu}(s)\, \underline{\nu}^T(t)}\, ds\, dt = \int_{t_0}^f \int_{t_0}^s \Phi(t, s)\, U(s)\, \overline{\underline{\nu}(s)\, \underline{\nu}^T(t)}\, dt\ \ ds$$

and interchanging the dummy variables s and t results in the identity

$$\int_{t_0}^f \int_t^f \Phi(t, s)\, U(s)\, \overline{\underline{\nu}(s)\, \underline{\nu}^T(t)}\, ds\, dt = \int_{t_0}^f \int_{t_0}^t \Phi(s, t)\, U(t)\, \overline{\underline{\nu}(s)\, \underline{\nu}^T(t)}\, ds\, dt \tag{4-26}$$

Combining Eqs. (4-25) and (4-26) yields

$$\begin{aligned}
\int_{t_0}^f \overline{\underline{\nu}^T(t)\, \underline{\nu}(t)}\, dt &= \mathrm{tr} \left\{ \int_{t_0}^f \int_t^f \Phi(t, s)\, U(s)\, \overline{\underline{\nu}(s)\, \underline{\nu}^T(t)}\, ds\, dt \right. \\
&\qquad \left. + \int_{t_0}^f \int_{t_0}^t \Phi(s, t)\, U(t)\, \overline{\underline{\nu}(s)\, \underline{\nu}^T(t)}\, ds\, dt \right\} \\
&= \mathrm{tr} \left\{ \int_{t_0}^f \int_{t_0}^f K(t, s)\, \overline{\nu(s)\, \nu^T(t)}\, ds\, dt \right\}
\end{aligned}$$

where

$$K(t, s) = \begin{cases} \Phi(t, s)\, U(s) & \text{if } t \le s \\ \Phi(s, t)\, U(t) & \text{if } t > s \end{cases}$$

Note that K(t, s) is continuous and that K(t, t) = U(t). Now, since $\overline{\underline{\nu}(s)\underline{\nu}^T(t)} = S(s)\,\delta\,(s - t)$, the equation above becomes

$$\int_{t_0}^{f} \overline{\underline{\nu}^T(t)\,\underline{\nu}(t)}\,dt = \text{tr}\left\{\int_{t_0}^{f}\int_{t_0}^{f} K(t, s)\,S(s)\,\delta\,(s - t)\,ds\,dt\right\}$$

$$= \int_{t_0}^{f} \text{tr}\left\{U(s)\,S(s)\right\}\,ds \tag{4-27}$$

Finally, combining Eqs. (4-23), (4-24), and (4-27) yields

$$\overline{w(t)} = \int_{t}^{f}\left\{\overline{\underline{\mu}^T A\underline{\mu}} + \text{tr}\,(U\,S)\right\}\,dt \tag{4-28}$$

and the result follows from Eqs. (4-21), (4-25) and the identity.

Section 5 The Separation Theorem

Theorem: Assume that the disturbing force $\underline{\nu}$ is white noise and is uncorrelated with the initial state of the dynamical system and the measurement noise $\underline{n}$. Then the control signal $\underline{u}$, which minimizes the mean mission cost, is

$$\underline{u}(t) = M(t)\,\tilde{L}_{MV}\left[\underline{m}\right](t) \tag{5-1}$$

where M(t) is the optimum feedback gain matrix, $M = -C_2^{-1}G^T U$, in the case when the state is known exactly, and $\tilde{L}_{MV}$ is the closed loop minimum variance estimator of the dynamical system state. If E(t) is the estimation error covariance matrix for L_{MV},

$$E = \overline{\left\{\underline{x} - L_{MV}\left[\underline{m}\right]\right\}\left\{\underline{x} - L_{MV}\left[\underline{m}\right]\right\}} \tag{5-2}$$

and s(t) is defined by Eq. (4-19), then the mean cost to complete the mission when the optimum feedback filter is employed, is given by the formula

$$\overline{C(t)} = \overline{\underline{x}^T(t)\,U(t)\,\underline{x}(t)} + \int_{t}^{f} \text{tr}\left\{M E M^T C_2 + S U\right\}\,dt \tag{5-1}$$

Proof: With any form of linear feedback, the dynamical system control signal u is the result of passing the measurements $\underline{m}$ through some linear filter $\tilde{L}$ so that $\underline{u} = \tilde{L}\left[\underline{m}\right]$. The deviation of the actual control signal $\underline{u}$ from the control signal $M\underline{x}$ which would be employed if the state was known exactly is

$$\underline{\mu} = \underline{\mu} - M\,\underline{x} = \tilde{L}\,[\underline{m}] - M\,\underline{x}$$

and, by Eq. (1-7)

$$\mu = \tilde{L}\,[H\,\underline{x}] - M\,\underline{x} + \tilde{L}\,[\underline{n}] \tag{5-2}$$

Also, by Eqs. (1-4) and (1-7) it follows that

$$\underline{x}' = F\,\underline{x} + G\tilde{L}\,[H\underline{x}] + G\tilde{L}\,[H\underline{n}] + \underline{\nu} \tag{5-3}$$

If follows by the hypothesis of the separation theorem and Eqs. (5-2) and (5-3) that $\underline{\mu}(t)$ and $\underline{x}(t)$ are uncorrelated with $\underline{\nu}(s)$ for $t \leq s$. Thus, Result 2 of Section 4 applies, and the mean mission cost is given by the formula

$$C = \overline{\underline{x}^T(t)\,U(t)\,\underline{x}(t)} + \int_0^f \left\{ \mathrm{Var}_{C_2} \left| \tilde{L}\,[\underline{m}] - M\,\underline{x} \right| + \mathrm{tr}(SU) \right\} dt \tag{5-4}$$

The design of the filter $\tilde{L}$ only affects the mean mission cost through the term $\mathrm{Var}_{C_2} \left| \tilde{L}\,[\underline{m}] - M\,\underline{x} \right|$ in Eq. (5-4). Obviously, the best choice for $\tilde{L}$ is the minimum variance closed loop estimator of $M\,\underline{x}$. By section 3, this estimator is $M\tilde{L}_{MV}$ where $\tilde{L}_{MV}$ is the closed loop minimum variance estimator of the dynamical system state $\underline{x}$. Thus, the first half of the theorem is proved.

Also by Result 2 of Section 4, the cost to complete the mission is given by the formula

$$C(t) = \overline{\underline{x}^T(t)\,U(t)\,\underline{x}(t)} + \int_t^f \mathrm{tr}\left\{ \overline{\underline{\mu}\underline{\mu}^T}\,C_2 + SU \right\} dt \tag{5-5}$$

But

$$\underline{\mu} = M \left\{ \tilde{L}_{MV}\,[\underline{m}] - \underline{x} \right\}$$

So

$$\overline{\underline{\mu}\underline{\mu}^T} = M\,\overline{\left\{ \tilde{L}_{MV}\,[\underline{m}] - \underline{x} \right\} \left\{ \tilde{L}_{MV}\,[\underline{m}] - \underline{x} \right\}^T} M^T$$

$$= M E M^T \tag{5-6}$$

and the second part of the theorem follows by substituting Eq. (5-6) into Eq. (5-5).

NOMENCLATURE

$\underline{a}$ A bar beneath a quantity denotes a vector. If a vector is used in an equation with matrices, it will be assumed to be a column vector.

NOMENCLATURE (cont)

$\bar{a}$ A bar over a quantity denotes the statistical mean or expected value of the quantity

M^T A superscript T attached to the matrix indicates the matrix transpose position.

x' A prime denotes differentiation with respect to time.

tr A Denotes the trace of the matrix A.

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